Section 11-3, Mathematics 104

Log Functions

A log function is an inverse exponential function.

Log functions are very difficult to understand and work with, unless you remember

A log function is an inverse exponential function

Let's see how this works.

Consider the exponential function $f(x) = 10^x$

x	$f(x)$
0	1
1	10
2	100
3	1000
-1	1/10
-2	1/100
-3	1/1000

The inverse of this function is written

$f^{-1}(x) = \log_{10} x$

x	$f^{-1}(x)$
1	0
10	1
100	2
1000	3
1/10	-1
1/100	-2
1/1000	-3

From this table, you should notice that a log is an exponent. This is something to keep in mind when dealing with logs.

Exponential function

Example:

$$f(x) = 2^x$$

$$f^{-1}(x) = \log_2 x$$

$$f(4) = 2^4 = 16$$

$$f^{-1}(16) = \log_2 16 = 4$$

Example:

$$f(x) = 16^x$$

$$f^{-1}(x) = \log_{16} x$$

$$f\left(\frac{1}{2}\right) = 16^{1/2} = \sqrt{16} = 4$$

$$f^{-1}(4) = \log_{16} 4 = \frac{1}{2}$$

Notice:

$f^{-1}(.001) = \log_{10} .001 = -3$ because $10^{-3} = .001$

So please remember, A LOG IS AN EXPONENT!

Here's what I mean by that.

$$\log_{10} 1000 = 3$$

So 3 is an exponent.

That is $10^3 = 1000$

Logs are one of the most confusing ideas we will learn.

That doesn't mean you have to be confused.

A Word on Notation

The following is included so to try to de-confuse you on notation.

For an exponential function a^x, the a is called the base.

We use the same name for log functions.

So we say $\log_{10} x$ as "Log to the base 10 of x"

There are two very common values you will see used as the bases for log functions.

They are 10 and e the Euler constant.

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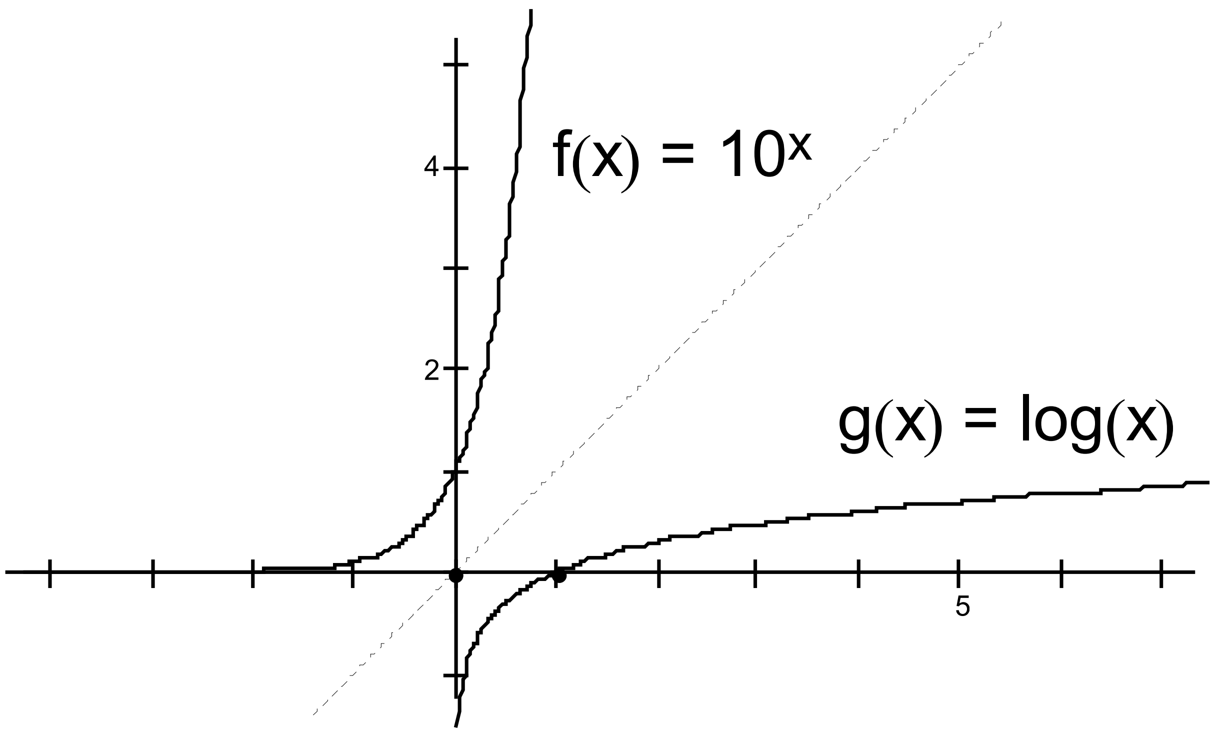
Graph of log functions

Since a log function is the inverse of an exponential function, you would expect they would be mirror images across y=x like all inverse functions.



While we have been only looking at integer values so far, the log function like the exponential function is a smooth continuous function so every value in the domain has a value in the range.

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Properties of Log Functions

All of the properties of log functions can be derived from the fact that

A log function is an inverse exponential function

1. $a^0 = 1$ $\qquad \log_a 1 = 0$
(Remember a is the base, 0 is a log so it is an exponent

2. $a^1 = a$ $\qquad \log_a a = 1$

3. $a^x = a^x$ $\qquad \log_a a^x = x$

Note that 3. is an expression of the property of inverse functions and the fact that a^x and $\log_a x$ are inverse functions. Number 4. is the other property of inverse functions.

4. $\qquad a^{\log_a x} = x$

Multiplying and Dividing by adding and multiplying Logs

5. $a^{x+y} = a^x \cdot a^y$ $\qquad \log_a x + \log_a y = \log_a xy$

6. $a^{x+y} = a^x \cdot a^y$ $\qquad \log_a x - \log_a y = \log_a \dfrac{x}{y}$

Stare at these two properties until they make sense.

Consider that

$\left(a^x\right)^2 = a^{2x}$ $\qquad \log_a x^2 = \log_a x + \log_a x = 2\log_a x$

So we get

7. $\left(a^x\right)^b = a^{bx}$ $\qquad \log_a x^b = b\log_a x$

Examples:

Calculating

$$\log_2 80 - \log_2 5 = \log_2 \frac{80}{5} = \log_2 16 = \log_2 2^4 = 4:$$

$$\log_4 2 + \log_4 32 = \log_4 32 \cdot 2 = \log_4 64 = \log_4 4^3 = 3$$

$$-\frac{1}{3}\log_{10} 8 = \log_{10} 8^{-1/3} = \log_{10} \frac{1}{8^{1/3}} = \log_{10} \frac{1}{\sqrt[3]{8}} = \log_{10} \frac{1}{2} = -.301$$

The last step requires a table or a calculator.

Expanding

$$\log_2 6x = \log_2 2 + \log_2 3 + \log_2 x = 1 + \log_2 3 + \log_2 x$$

$$\log_5 x^3 y^6 = \log_5 x^3 + \log_5 y^6 = 3\log_5 x + 6\log_5 y$$

$$\ln \frac{ab}{\sqrt[3]{c}} = \ln a + \ln b - \ln \sqrt[3]{c} = \ln a + \ln b - \ln c^{1/3} = \ln a + \ln b - \frac{\ln c}{3}$$

Combining

$$3\log x + \frac{1}{2}\log\left(x+1\right) = \log x^3 + \log\sqrt{x+1} = \log\left(x^3\sqrt{x+1}\right)$$

$$3\ln s + \frac{1}{2}\ln t - 4\ln\left(t^2+1\right) = \ln s^3 + \ln\sqrt{t} - \ln\left(t^2+1\right)^4 = \ln\left[\frac{s^3+\sqrt{t}}{\left(t^2+1\right)^4}\right]$$

Base 10 Logs

History Lesson

For many years before scientific calculators became available and inexpensive, scientists would use base 10 logarithms to do calculations. They would use **Log Tables**.

Log tables would have logs for the numbers $1 < x < 10$. This was called a **Mantissa.** The part of the log that represented a power of 10 was called the **Characteristic.**

Someone using the tables would need keep track of where the decimal point was.

Here's the process for multiplying two numbers.

1) Look up the first number in the tables and get the mantissa.
This could include doing a linear interpolation between two table values to get one more digit.

2) Add the characteristic.

3) Repeat for the second number

4) Add the two logs together (Remember adding logs is multiplying)

5) look up the new mantissa in the table in reverse, again interpolating to get the 5th digit.

6) Adjust the number's decimal point by the new characteristic.

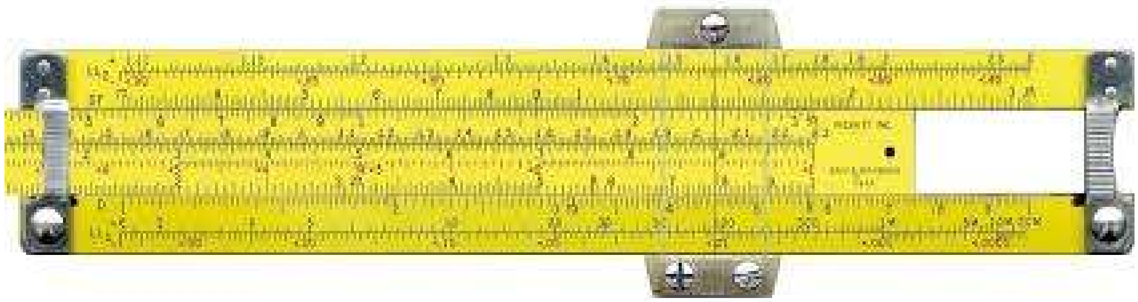
This was much faster than multiplying two 5 digits numbers, the alternative.

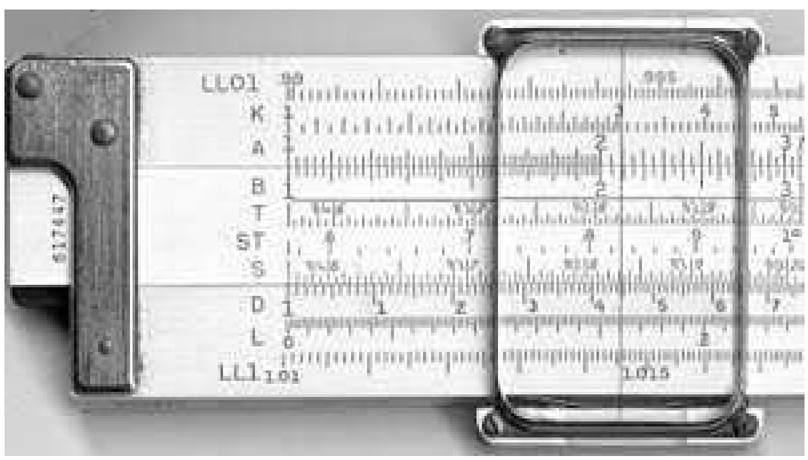
FIVE-PLACE MANTISSAS FOR COMMON LOGARITHMS

N.	0	1	2	3	4	5	6	7	8	9	Proportional parts			
100	00 000	043	087	130	173	217	260	303	346	389		**44**	**43**	**42**
101	432	475	518	561	604	647	689	732	775	817	1	4.4	4.3	4.2
102	860	903	945	988	*030	*072	*115	*157	*199	*242	2	8.8	8.6	8.4
103	01 284	326	368	410	452	494	536	578	620	662	3	13.2	12.9	12.6
104	703	745	787	828	870	912	953	995	*036	*078	4	17.6	17.2	16.8
105	02 119	160	202	243	284	325	366	407	449	490	5	22.0	21.5	21.0
106	531	572	612	653	694	735	776	816	857	898	6	26.4	25.8	25.2
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302	7	30.8	30.1	29.4
108	03 342	383	423	463	503	543	583	623	663	703	8	35.2	34.4	33.6
109	743	782	822	862	902	941	981	*021	*060	*100	9	39.6	38.7	37.8
110	04 139	179	218	258	297	336	376	415	454	493		**41**	**40**	**39**
111	532	571	610	650	689	727	766	805	844	883	1	4.1	4.0	3.9
112	922	961	999	*038	*077	*115	*154	*192	*231	*269	2	8.2	8.0	7.8
113	05 308	346	385	423	461	500	538	576	614	652	3	12.3	12.0	11.7
114	690	729	767	805	843	881	918	956	994	*032	4	16.4	16.0	15.6
115	06 070	108	145	183	221	258	296	333	371	408	5	20.5	20.0	19.5
116	446	483	521	558	595	633	670	707	744	781	6	24.6	24.0	23.4
117	819	856	893	930	967	*004	*041	*078	*115	*151	7	28.7	28.0	27.3
118	07 188	225	262	298	335	372	408	445	482	518	8	32.8	32.0	31.2
119	555	591	628	664	700	737	773	809	846	882	9	36.9	36.0	35.1
120	918	954	990	*027	*063	*099	*135	*171	*207	*243		**38**	**37**	**36**
121	08 279	314	350	386	422	458	493	529	565	600	1	3.8	3.7	3.6
122	636	672	707	743	778	814	849	884	920	955	2	7.6	7.4	7.2
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307	3	11.4	11.1	10.8
124	09 342	377	412	447	482	517	552	587	621	656	4	15.2	14.8	14.4
125	691	726	760	795	830	864	899	934	968	*003	5	19.0	18.5	18.0
126	10 037	072	106	140	175	209	243	278	312	346	6	22.8	22.2	21.6
127	380	415	449	483	517	551	585	619	653	687	7	26.6	25.9	25.2
128	721	755	789	823	857	890	924	958	992	*025	8	30.4	29.6	28.8
129	11 059	093	126	160	193	227	261	294	327	361	9	34.2	33.3	32.4
130	394	428	461	494	528	561	594	628	661	694		**35**	**34**	**33**
131	727	760	793	826	860	893	926	959	992	*024	1	3.5	3.4	3.3
132	12 057	090	123	156	189	222	254	287	320	352	2	7.0	6.8	6.6
133	385	418	450	483	516	548	581	613	646	678	3	10.5	10.2	9.9
134	710	743	775	808	840	872	905	937	969	*001	4	14.0	13.6	13.2
135	13 033	066	098	130	162	194	226	258	290	322	5	17.5	17.0	16.5
136	354	386	418	450	481	513	545	577	609	640	6	21.0	20.4	19.8
137	672	704	735	767	799	830	862	893	925	956	7	24.5	23.8	23.1
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270	8	28.0	27.2	26.4
139	14 301	333	364	395	426	457	489	520	551	582	9	31.5	30.6	29.7
140	613	644	675	706	737	768	799	829	860	891		**32**	**31**	**30**
141	922	953	983	*014	*045	*076	*106	*137	*168	*198	1	3.2	3.1	3.0
142	15 229	259	290	320	351	381	412	442	473	503	2	6.4	6.2	6.0
143	534	564	594	625	655	685	715	746	776	806	3	9.6	9.3	9.0
144	836	866	897	927	957	987	*017	*047	*077	*107	4	12.8	12.4	12.0
145	16 137	167	197	227	256	286	316	346	376	406	5	16.0	15.5	15.0
146	435	465	495	524	554	584	613	643	673	702	6	19.2	18.6	18.0
147	732	761	791	820	850	879	909	938	967	997	7	22.4	21.7	21.0
148	17 026	056	085	114	143	173	202	231	260	289	8	25.6	24.8	24.0
149	319	348	377	406	435	464	493	522	551	580	9	28.8	27.9	27.0
150	609	638	667	696	725	754	782	811	840	869				
N.	0	1	2	3	4	5	6	7	8	9	Proportional parts			

Note this is 1 of 20 pages one might use. The last column "Proportional Parts" was used to help with the linear interpolation.

For less accurate but quicker results, scientists and students would use a **Slide Ruler**.





The different lines on the ruler, eg. A, B, K, D, L are because the ruler could do other calculations such as square and cubed roots.

A slider ruler could cost anywhere from $5 for a cheap plastic one to over a hundred dollars for a fancy aluminum model.

A student carrying a slide ruler around was a sure sign that he was a science nerd.

This was when being a nerd was considered a very bad thing.

The Change of Base Formula

This is the hardest thing to remember with logs.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example:

Evaluate $\log_8 5$

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} = .77398$$